LIMIT POINTS OF A 2-BAR-TRUSS

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We dedicate this study to the Rector of the Volgograd State University of Architecture and Civil Engineering, Prof. Dr. Vladimir Ignatiev. The investigation of the bifurcation points of 2-bar-trusses which he presented in his paper "Stability Analysis of Trusses with the Principle of Virtual Displacements" has motivated us to undertake this study.

1. Task

Figure 1 shows a 2-bar-truss ACB which is symmetric with respect to axis x_2. Let the modulus of elasticity of the bars be E and the area of their cross-section A. A load P acting in the direction of the global axis x_2 displaces the apex C of the truss to D. The coordinates u_1 and u_2 of the displacement of node C are to be determined. It is also of interest to determine whether there exist values of P for which there is more than one equilibrium configuration of the truss. It is assumed that the strain in the bars is small and that the elastic limit of the material is not exceeded.

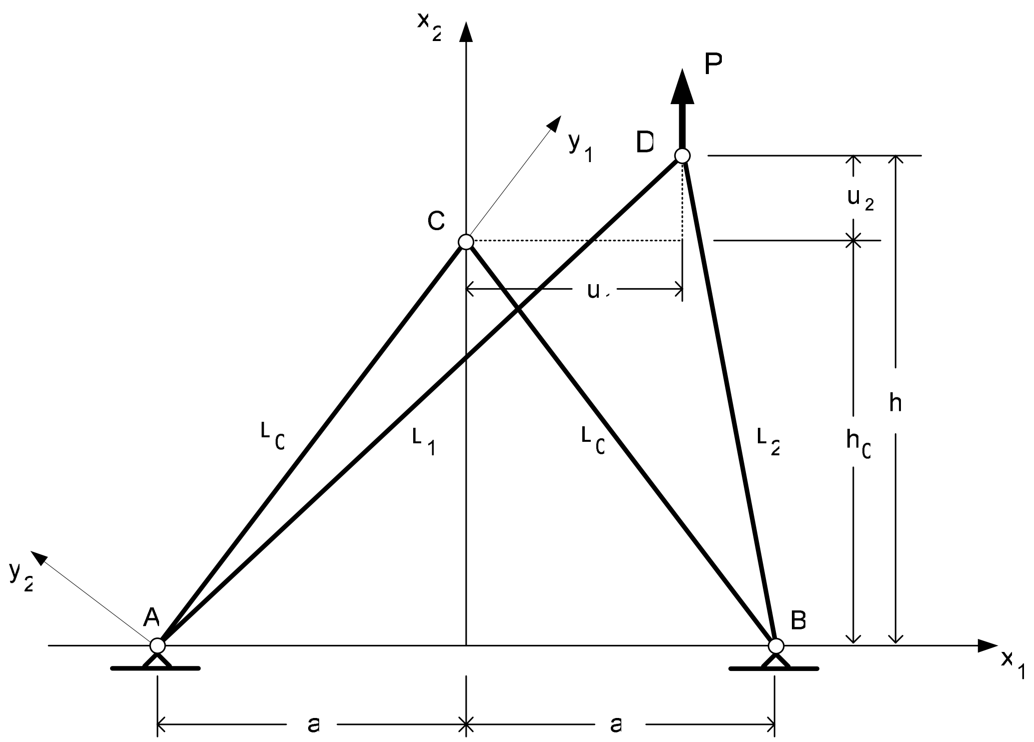

Figure 1: Displacement of a 2-Bar-Truss under a vertical load P

2. Kinematics of the Truss

A local coordinate system with origin A is defined for the reference configuration of the truss. The base vector $\mathbf{e}_1$ for axis y_1 points from A to C and has unit length. The base vector $\mathbf{e}_2$ for axis y_2 is orthogonal to $\mathbf{e}_1$ and also has unit length. The base vector $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ points in the direction of the global axis x_3. The coordinates in the local space of the displacement of a point on the axis are denoted by v_1 und v_2.

The axial strain of bar AD in the instant configuration of the truss is determined with the expressions for the well-known strain tensor of Green, as defined in the nonlinear theory of elasticity, and denoted by ε_1.

$$\mathbf{e}_1 = \frac{d\mathbf{y}}{dy_1} \tag{1}$$

$$\mathbf{g}_1 = \frac{d(\mathbf{y}+\mathbf{v})}{dy_1} = \mathbf{e}_1 + v_{1,1}\mathbf{e}_1 + v_{2,1}\mathbf{e}_2 \tag{2}$$

$$\varepsilon_1 = \frac{1}{2}(\mathbf{g}_1^T\mathbf{g}_1 - \mathbf{e}_1^T\mathbf{e}_1) = \frac{1}{2}((1+v_{1,1})^2 + (v_{2,1})^2 - 1) = \frac{1}{2}((\frac{L_1}{L_0})^2 - 1) \tag{3}$$

The lengths L_0 and L_1 follow from the geometry of the truss in Fig. 1:

$$\varepsilon_1 = \frac{(a+u_1)^2 + (h_0+u_2)^2 - (a^2+h_0^2)}{2L_0^2}$$

$$\varepsilon_1 = \frac{1}{2}\left(\frac{h_0}{L_0}\right)^2 \left(\frac{u_1}{h_0}(2\frac{a}{h_0}+\frac{u_1}{h_0}) + \frac{u_2}{h_0}(2+\frac{u_2}{h_0})\right) \tag{4}$$

The axial strain ε_2 of bar BD in the instant configuration of the truss is determined in an analogous manner:

$$\varepsilon_2 = \frac{1}{2}\left(\frac{h_0}{L_0}\right)^2 \left(\frac{u_1}{h_0}(-2\frac{a}{h_0}+\frac{u_1}{h_0}) + \frac{u_2}{h_0}(2+\frac{u_2}{h_0})\right) \tag{5}$$

2. Constitutive Law

In the linear theory of trusses, the technical stress σ in a bar is determined as the product of the Green strain ε with the modulus of elasticity E of the material of the bar. The technical stress is the ratio of the bar force N to the area A of the cross-section.

$$\sigma = E\varepsilon \tag{6}$$

$$\sigma = \frac{N}{A} \tag{7}$$

For the work that is done by the inner forces during nonlinear deformation of the continuum, it is shown in the theory of elasticity [1] that the technical stress is not conjugate to the Green strain ε, but that the 2. Piola-Kirchhoff stress tensor s is conjugate to ε. In the definition of this tensor, it is taken into account that the material points of base vector $\mathbf{e}_1$ of the reference configuration of bar AC are displaced to a base vector $\mathbf{b}_1$ in the direction of the bar axis AD in the instant configuration, whose length in general is not 1. The technical stress σ in bar AD in the instant configuration is referred to a vector of unit length in the direction of $\mathbf{b}_1$, the coordinate s of the 2. Piola-Kirchhoff stress tensor s to the base vector $\mathbf{b}_1$. Since the ratio of the lengths of the base vectors $\mathbf{e}_1$ and $\mathbf{b}_1$ equals the ratio of the reference length L_0 to the instant length L_1 of the bar, the stresses σ and s also have this ratio.

The relationship between the 2. Piola-Kirchhoff stress tensor **S** and the strain tensor **E** of Green is treated in the literature for both elastic and plastic material behaviour (see for instance chapter 5 in [2]). For elastic engineering materials, the linear constitutive law (8) with constant modulus of elasticity is used, which for linear behaviour reduces to (6) (see equations (5.4.9) in [2], (9.4.2) in [3] and (3.118) in [4]).

$$s = E\varepsilon \tag{8}$$

In the nonlinear theory, the axial force N in a bar is therefore determined as follows from the strain ε:

$$N = A\sigma = A\frac{L}{L_0}s = AE\frac{L}{L_0}\varepsilon \tag{9}$$

3. Statics of the Truss

The normal force n_1 in bar AD is determined from the strain ε_1 of the bar with (9):

$$n_1 = A\sigma_1 = AE\frac{L_1}{L_0}\varepsilon_1 \tag{10}$$

The normal force n_2 in bar BD is determined analogously:

$$n_2 = A\sigma_2 = AE\frac{L_2}{L_0}\varepsilon_2 \tag{11}$$

The forces which act at node D in the instant configuration are in equilibrium:

$$n_1\frac{h}{L_1} + n_2\frac{h}{L_2} = P \tag{12}$$

$$n_1\frac{a+u_1}{L_1} = n_2\frac{a-u_1}{L_1} \tag{13}$$

Substitution of (4), (5), (7) and (9) as well as $h = h_0 + u_2$ into (10) leads to:

$$P = AE\left(\frac{h_0}{L_0}\right)^3\left(1+\frac{u_2}{h_0}\right)\left((\frac{u_1}{h_0})^2 + (\frac{u_2}{h_0})^2 + 2(\frac{u_2}{h_0})\right) \tag{14}$$

The second equation for the determination of the unknowns u_1 and u_2 follows by substitution of (4), (5), (10) and (11) into (13):

$$\frac{u_1}{h_0}\left((\frac{u_1}{h_0})^2 + (\frac{u_2}{h_0})^2 + 2(\frac{u_2}{h_0}) + 2(\frac{a}{h_0})^2\right) = 0 \tag{15}$$

Equations (14) and (15) are solved for u_1 and u_2.

3. Solutions of the Governing Equations

Equation (15) shows that the truss can have two equilibrium configurations for a given load P, for which either the first factor or the second factor on the left-hand side of the equaqtion is null:

Solution 1: $u_1 = 0$

The trial function $u_1 = 0$ satisfies (15). The load $P(u_2)$ follows from (14):

$$P = AE\left(\frac{h_0}{L_0}\right)^3\left(\frac{u_2}{h_0}\right)\left(1+\frac{u_2}{h_0}\right)\left(2+\frac{u_2}{h_0}\right) \tag{16}$$

The normal force $N = n_1 = n_2$ in bars AD and BD follows from (4) and (10) with $u_1 = 0$:

$$N = \frac{AE}{2}\left(\frac{h_0}{L_0}\right)^3\left(\frac{u_2}{h_0}\right)\left(2+\frac{u_2}{h_0}\right)\sqrt{\left(\frac{a}{h_0}\right)^2+\left(1+\frac{u_2}{h_0}\right)^2} \tag{17}$$

The reaction at node A is in equilibrium with the bar force N. Its component in the direction of axis x_1 is denoted by R_1:

$$R_1 = -\frac{AE}{2}\left(\frac{h_0}{L_0}\right)^2\left(\frac{a}{L_0}\right)\left(\frac{u_2}{h_0}\right)\left(2+\frac{u_2}{h_0}\right) \tag{18}$$

A given value of P can be associated with several values of the displacement u_2. The behaviour of the truss is described with the following normalized variables:

normalized load $\quad p := \frac{P}{AE}\left(\frac{L_0}{h_0}\right)^3$ (19)

normalized displacement $\quad w_2 := \frac{u_2}{h_0}$ (20)

normalized reaction $\quad r := \frac{2R_1}{AE}\left(\frac{L_0}{h_0}\right)^2\left(\frac{L_0}{a}\right)$ (21)

Equations (16) and (18) are cast into normalized form:

$$p = w_2(1+w_2)(2+w_2) \tag{22}$$

$$r = -w_2(2+w_2) \tag{23}$$

The behaviour of the truss is shown in Fig.2. For the load range from p = -0.3849 to p = 0.3849 the truss has three equilibrium configurations. The three configurations for the load p = 0 are the reference configuration, a configuration in which the bars are collinear and a configuration which is the mirror image of the reference configuration. In the range -2.0 ≤ s ≤ 0.0 the load has extreme values p = -0.3849 at s = -0.44265 and p = 0.3849 at s = -1.57735.

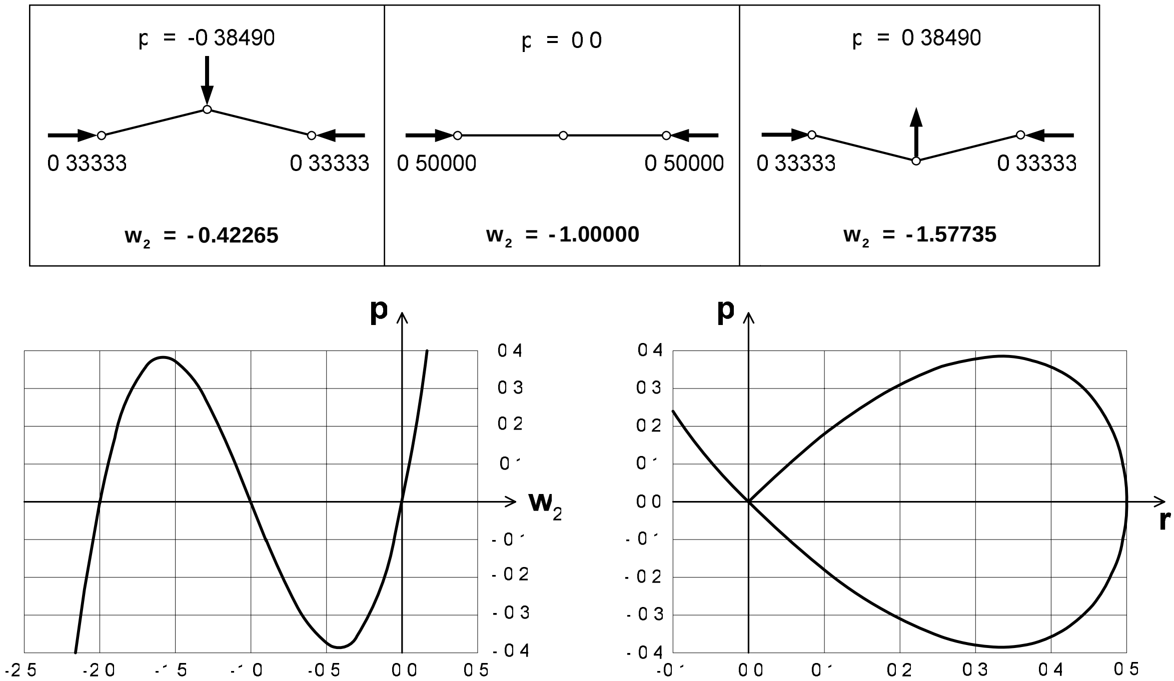

Fig 2: Displacement and reaction of a 2-bar-truss with $u_1 = 0$

The algebraic values of the load and the horizontal reaction vary as follows with the displacement of the truss:

$s > -0.42265$	:	decreasing load	increasing reaction
$-0.42265 \geq s > -1.00000$	:	increasing load	increasing reaction
$-1.00000 \geq s > -1.57735$	:	increasing load	decreasing reaction
$-1.57735 \geq s > -2.0000$	:	decreasing load	decreasing reaction
$-2.00000 \leq s$	:	decreasing load	decreasing reaction

Solution 2: $u_1 \neq 0$

The displacement u_1 is chosen so that the second factor on the left-hand side of (15) becomes null:

$$\left(\frac{u_1}{h_0}\right)^2 = -\left(\frac{u_2}{h_0}\right)^2 - 2\left(\frac{u_2}{h_0}\right) - 2\left(\frac{a}{h_0}\right)^2 \qquad (24)$$

The displacement u_1 is substituted from (24) into expression (14) for the load:

$$P = -2AE\left(\frac{h_0}{L_0}\right)\left(\frac{a}{L_0}\right)^2\left(1+\frac{u_2}{h_0}\right) \qquad (25)$$

A parameter m is defined for the geometry of the truss. The load and the displacements are normalised:

$$m = 2\left(\frac{a}{h_0}\right)^2 \qquad (26)$$

$$w_1 = \frac{u_1}{h_0} \qquad (27)$$

$$w_2 = \frac{u_2}{h_0} \qquad (28)$$

$$q = \frac{P}{2AE}\frac{L_0}{h_0}\left(\frac{L_0}{a}\right)^2 = \frac{p}{m} \qquad (29)$$

Substitution of (26) to (29) in to (24) and (25) leads to equations for w_1 and w_2

$$(w_1)^2 = -w_2(2+w_2) - m \qquad (30)$$

$$w_2 = -q-1 \qquad (31)$$

Substitution of (31) in to (30) leads to the quadratic equation (32) in w_1, which has real roots if condition (33) is satisfied.

$$(w_1)^2 = 1-m-q^2 \qquad (32)$$

$$q^2 \leq 1-m \qquad (33)$$

The load q is real if contition (34) is satisfied. The aspect ratio a = 0.7071 h_0 corresponds to an angle of 45 degrees between the axes x_1 and y_1.

$$m \leq 1 \quad \rightarrow \quad \frac{a}{h_0} \leq 0.7071 \qquad (34)$$

For a given value of m which satisfies condition (34), there are maximal loads q_{c1} und q_{c2} which satisfy condition (33):

$$q_{c1} = -\sqrt{1-m} \qquad (35)$$

$$q_{c2} = \sqrt{1-m} \qquad (36)$$

Fig. 3 shows the absolute value of the bifurcation load as a function of the aspect ratio of the truss.

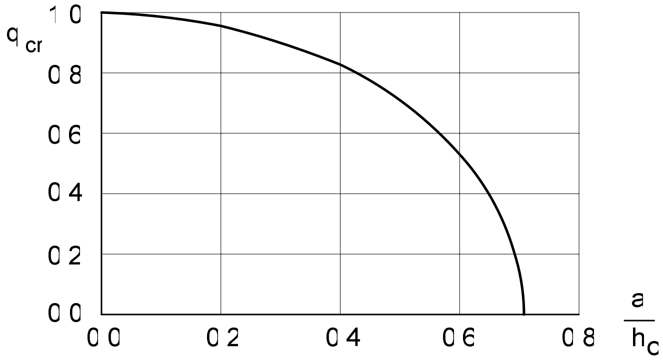

Figure 3 : Variation of the bifurcation load $|q_{cr}|$ with the aspect ratio $\frac{a}{h_0}$

To each load $q_{c1} \leq q \leq q_{c2}$ there corresponds a displacement state (w1,w2) which is determined with (31) and (32). The locus of the displacement of the apex of the truss is a circle with center (x_1,x_2) = (0,0) und radius $\sqrt{1-m}$:

$$(w_1)^2 + (1+w_2)^2 = 1-m \qquad (37)$$

Fig. 4 shows the load bearing behaviour of a 2-bar-truss with aspect ratio $a = 0.3h_0$. The load is null at point C. As the absolute value of the negative load increases, point C is displaced to D_1. This path is determined with solution 1, so that w_1 is null and the displacement w_2 follows from (22) and (29). The bifurcation load q_{c1} is reached at point D_1. As the absolute value of the lateral displacement w_1 increases, the absolute value of the load decreases and reaches null at point D_2 for $w_2 = 0$. If the load is then increased, the absolute value of the lateral displacement w_1 decreases and reaches null at point D_3 for the load q_{c2}. If the load is then decreased, the path is determined by solution 1 with $w_1 = 0$: the displacement w_2 is determined with (22) and (29). At point D_4 with location $(0,-h_0)$ the load is again null. Point D_4 is the mirror image of point C. Fig.5 shows the relationship between the normalised load q and the normalised displacement w_2 for the specified load path.

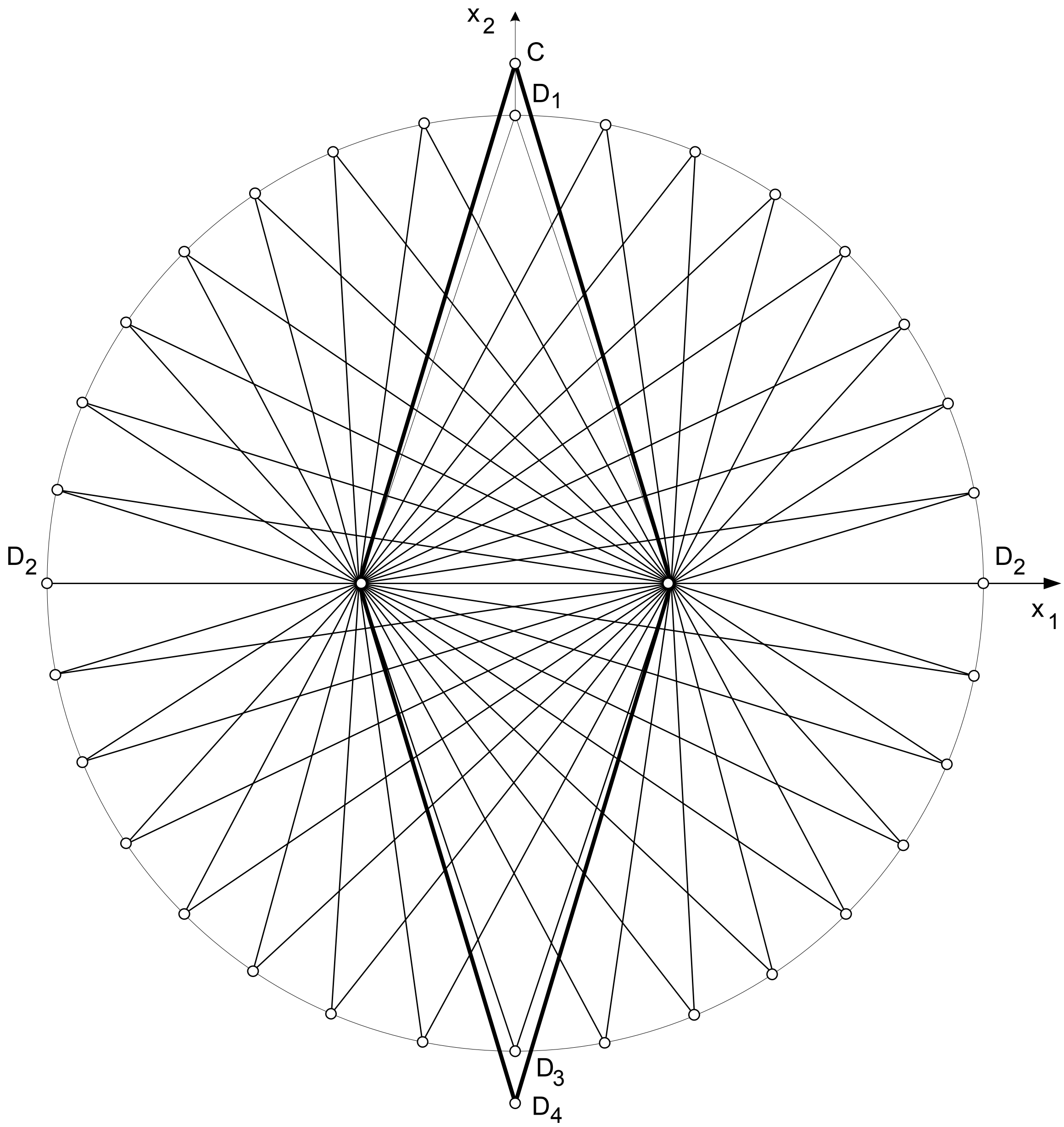

Figure 4 : Displacement states of a truss with aspect ratio $a = 0.3\ h_0$

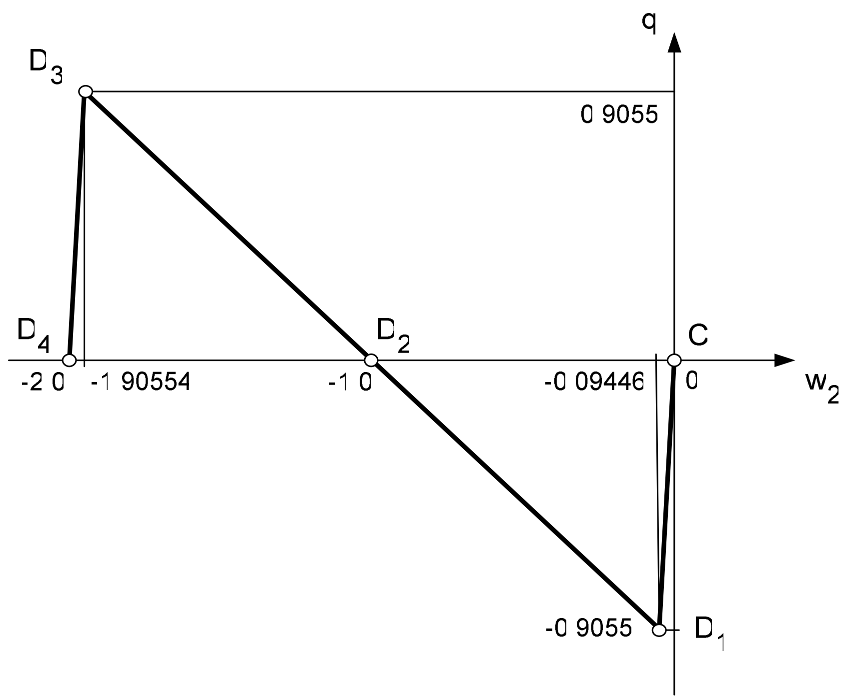

Figure 5: Load q as function of the displacement w_2 of node C in Fig. 4

5. Significance of the Yield Strain of the Material

The extremal axial strain ε_c in the bars of the truss for solution 1 is determined with equations (10) and (17):

$$\varepsilon_c = -\frac{1}{2}\left(\frac{h_0}{L_0}\right)^2 \tag{38}$$

The aspect ratio $h_0 = 0.2\ L_0$ leads to the strain $\varepsilon_c = -0.020$. For solution 2, the extremal strain corresponding to the bifurcation load is determined with (4):

$$\varepsilon_c = -\frac{1}{4}\left(\frac{a}{L_0}\right)^2 \tag{39}$$

The aspect ratio $a = 0.2\ L_0$ leads to the strain $\varepsilon_c = -\ 0.010$. The yield strain of common engineering materials is considerable smaller. For structural steel, it has the order of magnitude 0.0012. The geometric parameters from (38) und (39) which correspond to this strain are:

solution 1: $h_0 = 0.04899\ L_0$ (40)

solution 2: $a = 0.06928\ L_0$ (41)

The aspect ratio (40) corresponds to an angle of inclination of the bar axis of 2.81 degrees, thus an angle change in direction of 5.62 degrees at the apex of the truss. The aspect ratio (41) corresponds to an angle of inclination of the bar axis of 86.03 degrees, this a tip angle of 7.95 Grad. For larger aspect ratios, the behaviour of the truss contrary to the assumptions of the theory is no longer elastic.

6. Limit Points of the Truss

Points on the displacement-load diagram of a truss which correspond to unstable configurations of the truss are called limit points of the diagram or limit points of (the behaviour of) the truss.

If the ratio of the half base a to the height h_0 of the truss is less than or equal 0.7071, the truss has a limit point which is a bifurcation point. A load in the direction of axis x_2 which is less than the bifurcation load causes a displacement in the same direction. After the bifurcation point is reached, the load causes a displacement in the direction x_1 although it continues to act in the direction of axis x_2. The truss is unstable at the bifurcation point: an incremental displacement in the direction of axis x_1 takes place at constant load. If the aspect ratio exceeds 0.7071, the truss does not have a bifurcation point.

If the ratio of the height h_0 to the half base width a of the truss is small, the truss has a snap-through point as limit point. For sufficiently small values of h_0 to a the axial strain in the bars lies in the linear elastic range of the material, so that the assumptions of the theory are satisfied. The directions of the load and of the displacement at node C do not change at the snap-through point. This is in contrast with a bifurcation point, where the direction of the displacement changes. At the snap-through point, the truss is unstable, since an incremental displacement occurs in the direction of axis x_2 without a change in the load.

Since the truss is unstable at the limit points, the increment of its displacement at the limit point is an eigenform of its instant configuration. The associated eigenvalue is null. The eigenform for a bifurcation point differs from the eigenform for a snap-through point. At a bifurcation point, the eigenvector is orthogonal to the load vector. The eigenform at a snap-through point does not have this property.

For larger space trusses, an analytic determination of the limit points as in the present example is usually not possible. Therefore methods are being investigated which are derived from the general nonlinear theory of elasticity and permit a numerical determination of limit points for complex space trusses.

References

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